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Title: Compatibility Conditions for Quasisymmetry

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Compatibility Conditions for Quasisymmetry

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Quasisymmetry requires equilibrium **and** GC integrability

Definition 1. (quasisymmetric magnetic field)

A magnetic field $\boldsymbol{B}$ is *quasisymmetric* if

$$(\nabla \times \boldsymbol{B}) \times \boldsymbol{B} = \nabla \mathrm{p}$$

for some function p **and** the (leading-order) guiding center Lagrangian

$$\begin{aligned} L(\boldsymbol{X}, v_{\|}, \dot{\boldsymbol{X}}, \dot{v}_{\|}) =& (m v_{\|} \boldsymbol{b}(\boldsymbol{X}) + e\boldsymbol{A}(\boldsymbol{X})) \cdot \dot{\boldsymbol{X}} \\ & - \left(\frac{1}{2} m v_{\|}^2 + \mu |\boldsymbol{B}(\boldsymbol{X})|\right) \end{aligned}$$

admits a spatial symmetry.

There are various reasons to relax equilibrium constraint.

- Equilibria may support flow

$$\nabla \times (\boldsymbol{v} \times \boldsymbol{B}) = 0$$
$$\nabla \cdot (\rho \boldsymbol{v}) = 0$$
$$\rho \boldsymbol{v} \cdot \nabla \boldsymbol{v} + \nabla \mathrm{p} = (\nabla \times \boldsymbol{B}) \times \boldsymbol{B}$$

There are various reasons to relax equilibrium constraint.

- Equilibria sometimes support flow
- Equilibria sometimes support an anisotropic pressure tensor

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There are various reasons to relax equilibrium constraint.

- Equilibria sometimes support flow
- Equilibria sometimes support an anisotropic pressure tensor
- Active injection of particles or waves may alter equilibrium force balance

$$\nabla \times (\boldsymbol{v} \times \boldsymbol{B}) = 0$$

$$\nabla \cdot (\rho \boldsymbol{v}) = 0$$

$$\rho \boldsymbol{v} \cdot \nabla \boldsymbol{v} + \nabla \cdot \mathbb{P} = (\nabla \times \boldsymbol{B}) \times \boldsymbol{B} + \boldsymbol{F}_{\mathrm{ext}}$$

Weak quasisymmetry requires GC integrability only

Definition 2. (weakly quasisymmetric magnetic field)

A magnetic field $\boldsymbol{B}$ is *weakly quasisymmetric* if the (leading-order) guiding center Lagrangian

$$L(\boldsymbol{X}, v_\parallel, \dot{\boldsymbol{X}}, \dot{v}_\parallel) = (m v_\parallel \boldsymbol{b}(\boldsymbol{X}) + e\boldsymbol{A}(\boldsymbol{X})) \cdot \dot{\boldsymbol{X}} - \left(\frac{1}{2} m v_\parallel^2 + \mu |\boldsymbol{B}(\boldsymbol{X})|\right)$$

admits a spatial symmetry.

We know a little bit about weak quasisymmetry.

weak quasisymmetry: what we know

1. Quasisymmetry $\Rightarrow$ Weak Quasisymmetry
2. Weak Quasisymmetry $\not\Rightarrow$ Quasisymmetry
   - think of non-equilibrium axisymmetric fields

We mostly do not understand weak quasisymmetry.

weak quasisymmetry: open questions

1. Are all weakly-quasisymmetric $\boldsymbol{B}$ invariant under rotations and/or translations?

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2. If not, how to construct non-axisymmetric examples?

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3. Are quasisymmetric $\boldsymbol{B}$ perturbations of weakly-quasisymmetric $\boldsymbol{B}$?

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4. Is weak quasisymmetry weak enough to capture eqm flow or anisotropy?

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weak quasisymmetry: open questions

1. Are all weakly-quasisymmetric $\boldsymbol{B}$ invariant under rotations and/or translations?
2. If not, how to construct non-axisymmetric examples?
3. Are quasisymmetric $\boldsymbol{B}$ perturbations of weakly-quasisymmetric $\boldsymbol{B}$?
4. Is weak quasisymmetry weak enough to capture eqm flow or anisotropy?
5. Best ways to tackle (1)-(4)?

The purpose of this talk:

Show that weakly-quasisymmetric fields arise as solutions of a nonlinear PDE

Why?

- Provide a concrete description of weakly-quasisymmetric fields
  - the definition is rather abstract
- Introduce familiar framework to begin addressing open questions about weak quasisymmetry

The PDE will be derived as follows

Step One:

Identify PDE for generator $\boldsymbol{u}$ of weak quasisymmetry

Step Two:

Extract PDE for $\boldsymbol{B}$ as compatibility conditions for $\boldsymbol{u}$ equation to have solution

First step is straightforward Lagrangian mechanics

Theorem 1. (conditions for weak quasisymmetry)

A magnetic field $\boldsymbol{B}$ is weakly-quasisymmetric if and only if there is a vector field $\boldsymbol{u}$ such that

$$(\nabla \times \boldsymbol{B}) \times \boldsymbol{u} + \nabla(\boldsymbol{u} \cdot \boldsymbol{B}) = 0$$
$$\nabla \times (\boldsymbol{B} \times \boldsymbol{u}) = 0$$
$$\nabla \cdot \boldsymbol{u} = 0.$$

See:

- Burby, Qin, "Toroidal precession as a geometric phase," *Phys. Plasmas* **20**, 012511 (2013).
- R. S. MacKay's talk from first Simon's Meeting

Second step is more sublte

Basic idea:

Find conditions on B ensuring solution for $\boldsymbol{u}$ exists

Second step is more sublte

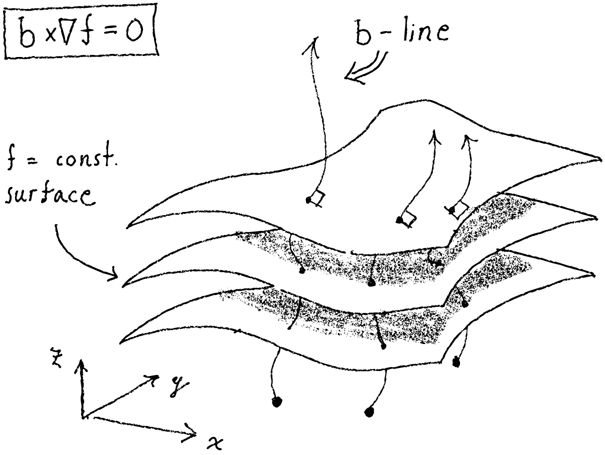
Basic idea:

Find conditions on B ensuring solution for $\boldsymbol{u}$ exists

How do we find such compatibility conditions?

Illustrative example

Surfaces perpendicular to a magnetic field



Perp. surfaces satisfy overdetermined PDE.

$$\boldsymbol{b}(\boldsymbol{x}) \times \nabla f(\boldsymbol{x}) = 0, \quad \forall \boldsymbol{x},\ \nabla f(\boldsymbol{x}) \neq 0$$

Three equations; one unknown function f!
$\Rightarrow$ We should expect compatibility conditions for a solution to exist

Equality of mixed partials gives compatibility condition.

$$0 = \nabla \cdot (\boldsymbol{b} \times \nabla f) = (\nabla \times \boldsymbol{b}) \cdot \nabla f$$

$\Downarrow$ (because a solution must satisfy $\nabla f = \lambda \boldsymbol{b}$) $\Downarrow$

$$\tau = \boldsymbol{b} \cdot \nabla \times \boldsymbol{b} = 0$$

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$$\tau = \boldsymbol{b} \cdot \nabla \times \boldsymbol{b} = 0$$

This is a compatibility condition!

Frobenius Thm says this is only compatibility condition.

Theorem (Frobenius)

a. If a solution f exists, $\tau = 0$ everywhere.
b. Conversely, if $\tau = 0$ everywhere, a smooth solution exists in a neighborhood of each $\boldsymbol{x}$.

Q: How can we generalize this example to treat weak quasisymmetry?

A: Resort to exterior differential systems (EDS) theory!

EDS theory systematizes finding **all*** compatibility conditions for **any** PDE.

CC extraction procedure:

(0) Reformulate PDE as vanishing of differential forms θ_i

$$\boldsymbol{b} \times \nabla f = 0$$
$$\Updownarrow$$
$$\theta = (\boldsymbol{b} \cdot d\boldsymbol{x}) \wedge df = 0$$

EDS theory systematizes finding **all*** compatibility conditions for **any** PDE.

CC extraction procedure:

(0) Reformulate PDE as vanishing of differential forms θ_i

(1) For each dependent variable Q, set $dQ = \boldsymbol{p}^{(Q)} \cdot d\boldsymbol{x}$ in $\theta_i, \mathbf{d}\theta_i$; find constraints on $\boldsymbol{p}^{(Q)}$'s. ($\boldsymbol{x}$ = independent variables)

$$\theta = (\boldsymbol{b} \cdot d\boldsymbol{x}) \wedge (\boldsymbol{p}^{(f)} \cdot d\boldsymbol{x}) = 0$$

$$\mathbf{d}\theta = (\nabla \times \boldsymbol{b} \cdot dS) \wedge (\boldsymbol{p}^{(f)} \cdot d\boldsymbol{x}) = 0$$

$$\Updownarrow$$

$$\boldsymbol{b} \times \boldsymbol{p}^{(f)} = 0$$

$$(\nabla \times \boldsymbol{b}) \cdot \boldsymbol{p}^{(f)} = 0$$

The $\boldsymbol{p}$'s may be interpreted as derivatives

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(2) If $\boldsymbol{p}$-equations constrain dependent variables, sub constraint into θ_i then return to (1). Otherwise move on.

f does not appear, so nothing to do!

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(2) If $\boldsymbol{p}$-equations constrain dependent variables, sub constraint into θ_i then return to (1). Otherwise move on.

(3) If $\boldsymbol{p}$-equations constrain independent variables, impose as CC

$$\boldsymbol{b} \times \boldsymbol{p}^{(f)} = 0, \quad (\nabla \times \boldsymbol{b}) \cdot \boldsymbol{p}^{(f)} = 0$$

$$\Updownarrow$$

$$\boldsymbol{p}^{(f)} = \lambda \boldsymbol{b}, \quad (\nabla \times \boldsymbol{b}) \cdot \boldsymbol{p}^{(f)} = 0$$

$$\Updownarrow$$

$$\boldsymbol{p}^{(f)} = \lambda \boldsymbol{b}, \quad \boldsymbol{b} \cdot (\nabla \times \boldsymbol{b})(\boldsymbol{x}) = 0$$

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(3) If $\boldsymbol{p}$-equations constrain independent variables, impose as CC

(4) If $\boldsymbol{p}$-equations solvable for each $\boldsymbol{x}$, apply *prolongation*. Otherwise return to (2).

$$\theta = (\boldsymbol{b} \cdot d\boldsymbol{x}) \wedge df \quad \rightarrow \quad \Theta = df - \lambda \boldsymbol{b} \cdot d\boldsymbol{x}$$

Prolongation differentiates the PDE in an intelligent way

EDS theory systematizes finding **all*** compatibility conditions for **any** PDE.

CC extraction procedure:

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(3) If $\boldsymbol{p}$-equations constrain independent variables, impose as CC

(4) If $\boldsymbol{p}$-equations solvable for each $\boldsymbol{x}$, apply *prolongation*. Otherwise return to (2).

(5) Apply "Cartan's test." If pass, stop. If fail, return to (1).

Cartan's test amounts to linear algebra. **Will not discuss here**.

EDS theory systematizes finding **all*** compatibility conditions for **any** PDE.

*Important Technicalities

- CCs necessary conditions for *any* solution to exist
- All CCs satisfied $\Rightarrow$ formal power series solutions exist
- If PDE coefficients real analytic, formal power series converge in small domain. (Generalized Cauchy-Kowalevski Thm.)
- Sometimes you're lucky! Satisfying all CCs may give exactly-solvable system.
    - This happens for weak quasisymmetry!

Preceding procedure produces all compatibility conditions for $\boldsymbol{u}$-equation.

Theorem 2. (compatibility conditions for weak quasisymmetry)

A non-vacuum magnetic field $\boldsymbol{B}$ is weakly-quasisymmetric if and only if there are potentials φ, ψ such that

$$\begin{aligned}
\nabla\varphi =& (\nabla \times \boldsymbol{B}) \times \boldsymbol{e} \\
\nabla\psi =& \boldsymbol{B} \times e^{-\varphi}\boldsymbol{e} \\
0 =& \boldsymbol{e} \cdot \nabla(\boldsymbol{B} \cdot \nabla|\boldsymbol{B}| \times \nabla\tau)
\end{aligned}$$

where $\tau = \boldsymbol{b} \cdot \nabla \times \boldsymbol{b}$ and the vector field $\boldsymbol{e}$ is given by

$$\boldsymbol{e} = \frac{\nabla|\boldsymbol{B}| \times \nabla\tau}{\boldsymbol{B} \cdot \nabla|\boldsymbol{B}| \times \nabla\tau}.$$

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$$\boldsymbol{e} =\frac{\nabla|\boldsymbol{B}| \times \nabla\tau}{\boldsymbol{B} \cdot \nabla|\boldsymbol{B}| \times \nabla\tau}.$$

Any $3D$ solution of this PDE will be non-axisymmetric weakly quasisymmetric field.

Where do we go from here?

1. Near-axis expansion of PDE for weak-quasisymmetry

Under consideration for publication in J. Plasma Phys.

Direct construction of optimized stellarator shapes. II. Numerical quasisymmetric solutions

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Quasisymmetric stellarators are appealing intellectually and as fusion reactor candidates since the guiding center particle trajectories and neoclassical transport are isomorphic to those in a tokamak, implying good confinement. Previously, quasisymmetric magnetic fields have been identified by applying black-box optimization algorithms to minimize symmetry-breaking Fourier modes of the field strength B. Here instead we directly construct magnetic fields in cylindrical coordinates that are quasisymmetric to leading order in distance from the magnetic axis, without using optimization. The method involves solution of a 1-dimensional nonlinear ordinary differential equation, originally derived by Garren and Boozer [*Phys. Fluids* B **3**, 2805 (1991)]. We demonstrate the usefulness and accuracy of this optimization-free approach by providing the results of this construction as input to the codes VMEC and BOOZ_XFORM, confirming the purity and scaling of the magnetic spectrum. The space of magnetic fields that are quasisymmetric to this order is parameterized by the magnetic axis shape along with three other real numbers, one of which reflects the on-axis toroidal current density, and another one of which is zero for stellarator symmetry. The method here could be used to generate good initial conditions for conventional optimization, and its speed enables exhaustive searches of parameter space.

2. **Even weaker** quasisymmetry

When $\epsilon = \rho/L = 0$, the GC Lagrangian blows up...

$$L(\boldsymbol{X}, v_\parallel, \dot{\boldsymbol{X}}, \dot{v}_\parallel) = \left(m v_\parallel \boldsymbol{b}(\boldsymbol{X}) + \frac{1}{\epsilon} e \boldsymbol{A}(\boldsymbol{X})\right) \cdot \dot{\boldsymbol{X}} - \left(\frac{1}{2} m v_\parallel^2 + \mu |\boldsymbol{B}(\boldsymbol{X})|\right)$$

Where do we go from here?

2. **Even weaker** quasisymmetry

...but the GC Poisson bracket and Hamiltonian do not.

$$\{f, g\} = (\boldsymbol{b} \cdot \nabla f)\partial_{v_\parallel} g - \partial_{v_\parallel} f(\boldsymbol{b} \cdot \nabla g) + O(\epsilon)$$

$$H = \frac{1}{2} m v_\parallel^2 + \mu |\boldsymbol{B}| + O(\epsilon)$$

Where do we go from here?

2. **Even weaker** quasisymmetry

Theorem 3 (Conditions for symmetry of leading-order Hamiltonian GC dynamics)

The leading-order GC Poisson bracket and Hamiltonian admit a spatial symmetry if and only if there is a $\boldsymbol{u}$ such that

$$\boldsymbol{u} \cdot \nabla |\boldsymbol{B}| = 0$$
$$\boldsymbol{u} \cdot \nabla \boldsymbol{B} - \boldsymbol{B} \cdot \nabla \boldsymbol{u} = 0.$$

(N. B. These conditions are satisfied automatically assuming weak quasisymmetry.)

What are the compatibility conditions on $\boldsymbol{B}$ to ensure $\boldsymbol{u}$ exists?

Where do we go from here?

3. Continue to improve understanding of weak quasisymmetry

- Switch roles of $\boldsymbol{B}$ and $\boldsymbol{u}$ to find compatibility conditions on $\boldsymbol{u}$.
  - e.g. if $\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^T = 0$ is compatibility condition, then *all weakly quasisymmetric fields must be axisymmetric*.

Where do we go from here?

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- If weak quasisymmetric fields exist, can they support flow or anisotropic pressure?

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- Switch roles of $\boldsymbol{B}$ and $\boldsymbol{u}$ to find compatibility conditions on $\boldsymbol{u}$.
  - e.g. if $\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^T = 0$ is compatibility condition, then *all weakly quasisymmetric fields must be axisymmetric*.
- If 3D weak quasisymmetric fields **possible**, can it support flow or anisotropic pressure?
- If 3D weak quasisymmetry **impossible**, how nearly can it be satisfied?

END